

**What Is Claimed Is:**

- 1           1.       A method for finding zeros of a function,  $f$ , within an interval,  $X$ ,  
2       using the interval version of Newton's method, wherein  $f'$  is the derivative of the  
3       function  $f$ , the method comprising:  
4           receiving a representation of the interval  $X$ , the representation including a  
5       first floating-point number,  $a$ , representing the left endpoint of  $X$ , and a second  
6       floating-point number,  $b$ , representing the right endpoint of  $X$ ;  
7           performing an interval Newton step on  $X$ , wherein the point of expansion  
8       is the midpoint,  $x$ , of the interval  $X$ , and wherein performing the interval Newton  
9       step involves evaluating  $f(x)$  to produce an interval result  $f^I(x)$ ; and  
10       if  $f^I(x)$  contains zero,  
11                 evaluating  $f(a)$  to produce an interval result  $f^I(a)$ ,  
12                 evaluating  $f(b)$  to produce an interval result  $f^I(b)$ ,  
13                 evaluating a termination condition for the processing of the  
14       current interval  $X$ , wherein the termination condition is TRUE if a  
15       number of conditions are satisfied, including if  $f^I(a)$  contains zero  
16       and if  $f^I(b)$  contains zero, and  
17                 if the termination condition is TRUE, terminating the  
18       processing of the current interval  $X$ , and recording  $X$  as a final  
19       bound.
- 1           2.       The method of claim 1, wherein if  $f^I(a)$  does not contain zero,  
2       evaluating  $f(a)$  additionally involves performing an interval Newton step wherein  
3       the point of expansion is  $a$ .

1           3.     The method of claim 1, wherein if  $f^I(b)$  does not contain zero,  
2     evaluating  $f(b)$  additionally involves performing an interval Newton step wherein  
3     the point of expansion is  $b$ .

1           4.     The method of claim 1, wherein if  $f^I(x)$  contains zero and  $f'(X)$   
2     contains zero, the termination condition for processing the current interval  $X$  is  
3     TRUE if  $f^I(a)$  contains zero,  $f^I(b)$  contains zero,  $f^I(x_1)$  contains zero and  $f^I(x_2)$   
4     contains zero;  
5           wherein  $x_1$  is the midpoint between  $a$  and  $x$ ; and  
6           wherein  $x_2$  is the midpoint between  $x$  and  $b$ .

1           5.     The method of claim 4, wherein if  $f^I(x)$  contains zero and if  $f'(X)$   
2     contains zero, and if either  $f^I(x_1)$  or  $f^I(x_2)$  does not contain zero, the method  
3     further comprises:  
4           splitting the interval  $X$  in half; and  
5           applying the interval Newton method to each half separately.

1           6.     The method of claim 1, wherein if  $f^I(x)$  does not contain zero and if  
2      $f'(X)$  contains zero, the method further comprises terminating the processing of  
3     the current interval  $X$ , and recording  $X$  as a final bound on condition that:  
4           the width of the interval  $X$  divided by the magnitude of the interval  $X$  is  
5           less than a first threshold value; and  
6           the magnitude of  $f(X)$  is less than a second threshold value.

1           7.     The method of claim 1, wherein if a given Newton step does not  
2     reduce the width of an interval by at least half, the method further comprises

3 splitting the interval in half and applying the interval Newton method to each of  
4 the two halves separately.

1 8. The method of claim 1, wherein if an interval Newton step results  
2 in two intervals, the method further comprises applying the interval Newton  
3 method to each of the two intervals separately.

1 9. The method of claim 1, wherein if the result of an interval Newton  
2 step is the empty interval, the method returns to process another interval.

1 10. The method of claim 1, wherein if  $f^I(a)$  contains zero and if  $f^I(b)$   
2 contains zero, the method returns to process another interval.

1 11. The method of claim 1,  
2 wherein if  $f^I(a)$  does not contain zero or if  $f^I(b)$  does not contain zero, the  
3 method further comprises performing an interval Newton step wherein the point  
4 of expansion is the midpoint of the interval  $X$ ; and

5 wherein if the result of the interval Newton step about the midpoint is the  
6 empty interval, the method returns to process another interval.

1 12. The method of claim 11, further comprising terminating the  
2 processing of the current interval  $X$  after a predetermined number of iterations.

1 13. A computer-readable storage medium storing instructions that  
2 when executed by a computer cause the computer to perform a method for finding  
3 zeros of a function,  $f$ , within an interval,  $X$ , using the interval version of Newton's  
4 method, wherein  $f'$  is the derivative of the function  $f$ , the method comprising:

5 receiving a representation of the interval  $X$ , the representation including a  
 6 first floating-point number,  $a$ , representing the left endpoint of  $X$ , and a second  
 7 floating-point number,  $b$ , representing the right endpoint of  $X$ ;  
 8 performing an interval Newton step on  $X$ , wherein the point of expansion  
 9 is the midpoint,  $x$ , of the interval  $X$ , and wherein performing the interval Newton  
 10 step involves evaluating  $f(x)$  to produce an interval result  $f^I(x)$ ; and  
 11 if  $f^I(x)$  contains zero,  
 12 evaluating  $f(a)$  to produce an interval result  $f^I(a)$ ,  
 13 evaluating  $f(b)$  to produce an interval result  $f^I(b)$ ,  
 14 evaluating a termination condition for the processing of the  
 15 current interval  $X$ , wherein the termination condition is TRUE if a  
 16 number of conditions are satisfied, including if  $f^I(a)$  contains zero  
 17 and if  $f^I(b)$  contains zero, and  
 18 if the termination condition is TRUE, terminating the  
 19 processing of the current interval  $X$ , and recording  $X$  as a final  
 20 bound.

1 14. The computer-readable storage medium of claim 13, wherein if  
 2  $f^I(a)$  does not contain zero evaluating  $f(a)$  additionally involves performing an  
 3 interval Newton step wherein the point of expansion is  $a$ .

1 15. The computer-readable storage medium of claim 13, wherein if  
 2  $f^I(b)$  does not contain zero, evaluating  $f(b)$  additionally involves performing an  
 3 interval Newton step wherein the point of expansion is  $b$ .

1 16. The computer-readable storage medium of claim 13, wherein if  
 2  $f^I(x)$  contains zero and  $f'(X)$  contains zero, the termination condition for

3 processing the current interval  $X$  is TRUE if  $f^I(a)$  contains zero,  $f^I(b)$  contains  
4 zero,  $f^I(x_1)$  contains zero and  $f^I(x_2)$  contains zero;  
5 wherein  $x_1$  is the midpoint between  $a$  and  $x$ ; and  
6 wherein  $x_2$  is the midpoint between  $x$  and  $b$ .

1 17. The computer-readable storage medium of claim 16, wherein if  
2  $f^I(x)$  contains zero and if  $f'(X)$  contains zero, and if either  $f^I(x_1)$  or  $f^I(x_2)$  does not  
3 contain zero, the method further comprises:  
4 splitting the interval  $X$  in half; and  
5 applying the interval Newton method to each half separately.

1 18. The computer-readable storage medium of claim 13, wherein if  
2  $f^I(x)$  does not contain zero and if  $f'(X)$  contains zero, the method further  
3 comprises terminating the processing of the current interval  $X$ , and recording  $X$  as  
4 a final bound on condition that:  
5 the width of the interval  $X$  divided by the magnitude of the interval  $X$  is  
6 less than a first threshold value; and  
7 the magnitude of  $f(X)$  is less than a second threshold value.

1 19. The computer-readable storage medium of claim 13, wherein if a  
2 given Newton step does not reduce the width of an interval by at least half, the  
3 method further comprises splitting the interval in half and applying the interval  
4 Newton method to each of the two halves separately.

1 20. The computer-readable storage medium if claim 13, wherein if an  
2 interval Newton step results in two intervals, the method further comprises  
3 applying the interval Newton method to each of the two intervals separately.

1           21.     The computer-readable storage medium of claim 13, wherein if the  
2     result of an interval Newton step is the empty interval, the method returns to  
3     process another interval.

1           22.     The computer-readable storage medium of claim 13, wherein if  
2      $f^I(a)$  contains zero and if  $f^I(b)$  contains zero, the method returns to process  
3     another interval.

1           23.     The computer-readable storage medium of claim 13,  
2             wherein if  $f^I(a)$  does not contain zero or if  $f^I(b)$  does not contain zero, the  
3     method further comprises performing an interval Newton step wherein the point  
4     of expansion is the midpoint of the interval  $X$ ; and  
5             wherein if the result of the interval Newton step about the midpoint is the  
6     empty interval, the method returns to process another interval.

1           24.     The computer-readable storage medium of claim 13, wherein the  
2     method further comprises terminating the processing of the current interval  $X$  after  
3     a predetermined number of iterations.

1           25.     An apparatus that finds zeros of a function,  $f$ , within an interval,  $X$ ,  
2     using the interval version of Newton's method, wherein  $f'$  is the derivative of the  
3     function  $f$ , the apparatus comprising:  
4             a receiving mechanism that is configured to receive a representation of the  
5     interval  $X$ , the representation including a first floating-point number,  $a$ ,  
6     representing the left endpoint of  $X$ , and a second floating-point number,  $b$ ,  
7     representing the right endpoint of  $X$ ;

8 an interval Newton mechanism that is configured to perform an interval  
9 Newton step on  $X$ , wherein the point of expansion is the midpoint,  $x$ , of the  
10 interval  $X$ , and wherein performing the interval Newton step involves evaluating  
11  $f(x)$  to produce an interval result  $f^I(x)$ ; and  
12 wherein if  $f^I(x)$  contains zero, the interval Newton mechanism is  
13 configured to,  
14 evaluate  $f(a)$  to produce an interval result  $f^I(a)$ ,  
15 evaluate  $f(b)$  to produce an interval result  $f^I(b)$ ,  
16 evaluate a termination condition for the processing of the  
17 current interval  $X$ , wherein the termination condition is TRUE if a  
18 number of conditions are satisfied, including if  $f^I(a)$  contains zero  
19 and if  $f^I(b)$  contains zero, and  
20 if the termination condition is TRUE, to terminate the  
21 processing of the current interval  $X$ , and to record  $X$  as a final  
22 bound.

1 26. The apparatus of claim 25, wherein while evaluating  $f(a)$  to  
2 produce the interval result  $f^I(a)$ , the interval Newton mechanism is additionally  
3 configured to perform an interval Newton step wherein the point of expansion is  $a$   
4 if  $f^I(a)$  does not contain zero.

1 27. The apparatus of claim 25, wherein while evaluating  $f(b)$  to  
2 produce the interval result  $f^I(b)$  the interval Newton mechanism is additionally  
3 configured to perform an interval Newton step wherein the point of expansion is  $b$   
4 if  $f^I(b)$  does not contain zero.

1           28.    The apparatus of claim 25, wherein if  $f^l(x)$  contains zero and  $f'(X)$   
2 contains zero, the termination condition for processing the current interval  $X$  is  
3 TRUE if  $f^l(a)$  contains zero,  $f^l(b)$  contains zero,  $f^l(x_1)$  contains zero and  $f^l(x_2)$   
4 contains zero;  
5           wherein  $x_1$  is the midpoint between  $a$  and  $x$ ; and  
6           wherein  $x_2$  is the midpoint between  $x$  and  $b$ .

1           29.    The apparatus of claim 28, wherein if  $f^l(x)$  contains zero and if  
2  $f'(X)$  contains zero, and if either  $f^l(x_1)$  or  $f^l(x_2)$  does not contain zero, the interval  
3 Newton mechanism is additionally configured to:  
4           split the interval  $X$  in half; and to  
5           apply the interval Newton method to each half separately.

1           30.    The apparatus of claim 25, wherein if  $f^l(x)$  does not contain zero  
2 and if  $f'(X)$  contains zero, the interval Newton mechanism is additionally  
3 configured to terminate the processing of the current interval  $X$ , and to record  $X$  as  
4 a final bound on condition that:  
5           the width of the interval  $X$  divided by the magnitude of the interval  $X$  is  
6 less than a first threshold value; and  
7           the magnitude of  $f(X)$  is less than a second threshold value.

1           31.    The apparatus of claim 25, wherein if a given Newton step does  
2 not reduce the width of an interval by at least half, the interval Newton  
3 mechanism is additionally configured to split the interval in half, and to apply the  
4 interval Newton method to each of the two halves separately.



1           32.     The apparatus of claim 25, wherein if an interval Newton step  
2 results in two intervals, the interval Newton mechanism is additionally configured  
3 to apply the interval Newton method to each of the two intervals separately.

1           33.     The apparatus of claim 25, wherein if the result of an interval  
2 Newton step is the empty interval, the interval Newton mechanism is additionally  
3 configured to return to process another interval.

1           34.     The apparatus of claim 25, wherein if  $f^l(a)$  contains zero and if  
2  $f^l(b)$  contains zero, the interval Newton mechanism is additionally configured to  
3 return to process another interval.

1           35.     The apparatus of claim 25,  
2 wherein if  $f^l(a)$  does not contain zero or if  $f^l(b)$  does not contain zero, the  
3 interval Newton mechanism is additionally configured to perform an interval  
4 Newton step wherein the point of expansion is the midpoint of the interval  $X$ ; and  
5 wherein if the result of the interval Newton step about the midpoint is the  
6 empty interval, the interval Newton mechanism is additionally configured to  
7 return to process another interval.

1           36.     The apparatus of claim 35, wherein the interval Newton  
2 mechanism is additionally configured to terminate the processing of the current  
3 interval  $X$  after a predetermined number of iterations.